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ARBITRARY MOTION OF AN ELLIPTIC DISC AT A FLUID INTERFACE

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Abstract—The problem of calculating the disturbance due to finite elliptic discs at the interface ($x_3 = 0$) of two incompressible immiscible fluids of different coefficients of viscosity is solved, assuming that body and inertia forces are negligible. When the direction of motion is parallel to the interface, our solution, which is based on potential functions analogous to the Papkovitch–Neuber functions of linear elasticity, satisfies not only the interface conditions of continuity of fluid velocity and stresses but also that of zero normal velocity at the interface. It is also remarkable that this solution produces in each of the fluids a flow field that is totally independent of the properties of the other fluid. These results are not peculiar to elliptic discs, but also hold for finite discs of other shapes. The method of solution presented here can be readily applied to the more general cases where the two-phase fluid, in the absence of the disc, moves with an arbitrarily directed velocity which is a general polynomial function of the coordinates x_1 and x_2 at the interface. The procedure for carrying this out is demonstrated by treating the case of an elliptic disc in linear shear flow.

1. INTRODUCTION

The fluid dynamics of two-phase fluids perturbed by bodies moving through such fluids has recently assumed considerable importance in chemical engineering and bio-medical engineering. In some of the applications in these fields of engineering, the problem is generally one of great complexity which can in many cases be reduced to a more manageable problem, without significantly altering the basic character and features of the flow, by making some plausible assumptions. In some of such cases the body is idealised to be a disc. Furthermore, the flow field produced by the motion of the disc may be assumed to be quasi-steady provided that a suitable defined Reynolds number (usually based on the disc's dimension) is small. This paper deals with the motion of such an idealised body of elliptic shape at the interface of two immiscible fluids.

The problem of the motion of a circular disc moving with uniform but arbitrarily directed velocity at the interface of two immiscible fluids was recently studied by Ranger (1978) and Olunloyo (1978). In both studies, the interface which is originally the plane $x_3 = 0$ is assumed to remain unaffected in its shape by the motion of the disc. Ranger (1978) solved the problem using the method of complementary integral representation at the interface. On the other hand, Olunloyo solved the same problem by using suitable integral representations of the general Papkovitch–Neuber and Boussinesq potential functions.

In two-phase low Reynolds number flows, the matter of the correct conditions that should be satisfied at the interface is yet to be settled. There are two schools of thought on this matter. One school (see for example Aderogba 1976) holds the view that the conditions of continuity of the tangential and normal components of stresses and velocities at the interface seem plausible, even though this may allow the interface to move normal to itself and possibly distort. The alternative view (which is the one supported by Lee *et al.* 1979, for example) is that the interface should be stationary while the normal stress component may vary discontinuously across the interface. The net force which may act on the interface in the latter case could either cause the interface to accelerate or be balanced by surface tension forces, which also cause the interface to distort.

However, for a flat disc in a steady stream of homogeneous fluid, a solution exists which yields zero values of stress components and normal velocity at all points in the plane of the disc except those on the disc itself provided the direction of the stream is parallel to the interface. In other words, the plane which contains the disc acts as a "free surface" bordered by a vacuum.

It follows therefore, that the solution to the corresponding problem of a disc moving at the interface of a two-phase fluid can be obtained by combining, *without any modification whatsoever*, the single-phase solutions for each of the phases. There is then no interaction between the phases and the two-phase problem reduces to the problem of finding the single phase solution which also yields zero stresses and normal velocity at the interface. If such solution exists, as it does in this case, the controversy about the interface conditions to apply does not arise. When the direction of motion of the stream is normal to the interface, our solution gives rise to zero stress components and continuous velocity components at the interface. Thus the solution is also useful in constructing solution to the problem of a disc moving normal to a fluid interface if the interface conditions are assumed to be those of continuity of stresses and velocities. In fact no solution exists for the case where the direction of motion of the disc is normal to the interface if the condition of zero normal interface velocity is enforced unless, as one may expect, the disc is subjected to an infinite force. This observation is corroborated by the finding of Ranger (1978).

The first part of this paper is devoted to developing the desired single-phase solution for an elliptic disc in a uniform, but arbitrarily directed flow. The method used is based on potential functions, akin to the Papkovitch–Neuber potentials of linear elasticity, and the properties of potentials due to elliptic laminae of variable surface densities. The second part of the paper treats the problem of an elliptic disc in linear shear flow. Finally, an outline is given of the procedure to be adopted in generalising this method to the case of a disc in a stream whose undisturbed velocity at the position of the disc is a general polynomial function of position.

2. GOVERNING EQUATIONS

If body and inertia forces are negligible, the equation of motion for an imcompressible Newtonian fluid of viscosity μ is

$$P_{,i} = \mu u_{i,jj}. \tag{1}$$

The equation of continuity for the same fluid is

$$u_{i,i} = 0. \tag{2}$$

In [1] and [2] P and u_i stand for the pressure and velocity vector respectively. The usual subscript notation is employed: subscripts preceded by comma denote differentiation with respect to the appropriate coordinate, e.g. $u_{i,3} = (\partial u_i/\partial x_3)$; repeated letter subscripts imply, unless otherwise stated, summation over the values 1, 2, 3. Note that [1] and [2] also hold approximately for unsteady flows if Reynolds number defined in terms of fluid density ρ, free stream velocity U_∞, largest disc dimension a_{max} and μ as

$$\text{Re} = \rho U_\infty a_{max}/\mu$$

is much smaller than unity. The stress components, σ_{ij}, are then given by

$$\sigma_{ij} = -P\delta_{ij} + \mu(u_{i,j} + u_{j,i}). \tag{3}$$

Aderogba (1974) showed that a complete general solution to [1] and [2], similar to the Papkovitch–Neuber solution to problems of elastic behaviour of isotropic materials, is

$$u_i = (\psi_0 + x_j\psi_j)_{,i} - 2\psi_i$$
$$P = 2\mu\psi_{i,i}, \tag{4}$$

where $\nabla^2\psi_0 = \nabla^2\psi_i = 0$ and ∇^2 is the Laplacian operator.

There is an infinite number of functions which satisfy these conditions but the relevant functions for any given problem are those that also satisfy the boundary conditions of the problem. In the case of a disc moving in a viscous fluid the solution is uniquely determined by requiring that the no-slip condition

$$u_i = 0 \qquad [5]$$

be satisfied on the disc. It is shown in the next section that when the undisturbed fluid velocity is uniform, the presence of an elliptic disc in the fluid induces zero stresses and continuous normal velocity in the plane of the disc except at points on the disc (where stresses may not be zero).

3. ELLIPTICAL DISC IN UNIFORM ONE-PHASE FLOW

Consider an elliptic disc with semi-axes a_1, a_2, lying in the plane $x_3 = 0$ with its centre at $(0, 0, 0)$. Let U_i be the uniform undisturbed velocity of a stream flowing past the disc. We seek potential function ψ_0, ψ_i ($i = 1, 2, 3$) such that the velocity vector u_i, when expressed as

$$u_i = U_i + (\psi_0 + x_j\psi_j)_{,i} - 2\psi_i,$$

satisfies the no-slip condition of [5] on the disc.

Guided by the properties of the potential field due to elliptic laminae of variable surface density and the requirements which the solution to the present problem has to satisfy, we select the following representation for ψ_0 and ψ_i.

$$\psi_0 = A_j \int\int_s [\xi_j\{1-(\xi_1/a_1)^2-(\xi_2/a_2)^2\}^{-1/2}/R]\,\mathrm{d}s \qquad j = 1, 2 \qquad [6]$$

$$\psi_i = B_i \int\int_s [\{1-(\xi_1/a_1)^2-(\xi_2/a_2)^2\}^{-1/2}/R]\,\mathrm{d}s \qquad i = 1, 2, 3 \qquad [7]$$

where R is the distance from a point $(\xi_1, \xi_2, 0)$ on the disc to the general point (x_1, x_2, x_3), i.e.

$$R = \{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2\}^{1/2}.$$

In [5] and [6] A_j and B_i are constant vectors. It can be deduced from the work of Dyson (1891) and Ferrers (1877) that

$$\left.\begin{aligned} \psi_0 &= \epsilon a_j^2 A_j x_j \int_\nu^\infty \{(a_j^2+\eta)Q(\eta)\}^{-1}\,\mathrm{d}\eta \qquad j = 1, 2 \\ &\text{and} \\ \psi_i &= \epsilon B_i \int_\nu^\infty Q^{-1}(\eta)\,\mathrm{d}\eta \qquad i = 1, 2, 3. \end{aligned}\right\} \qquad [8]$$

In [8], $\nu = 0$ for points on the disc while for points off the disc, ν is the least positive root of the equation

$$x_1^2/(a_1^2+\nu) + x_2^2/(a_2^2+\nu) + x_3^2/\nu = 0.$$

Also,

$$\epsilon = \pi a_1 a_2$$

and

$$Q(\eta) = \{a_1^2 + \eta)(a_2^2 + \eta)\eta\}^{1/2}.$$

We here point out certain properties of ν on the disc as these will be found useful later on in this paper. We first note that

$$\nu_{,i} = \frac{\partial \nu}{\partial x_i} = 2x_i/\{(a_i + \nu)[x_1^2/(a_1^2 + \nu)^2 + x_2^2/(a_2^2 + \nu)^2 + x_3^2/\nu^2]\}. \qquad [9]$$

where we have taken a_3 to be zero.

From [9] we deduce that as $\nu \to 0$,

$$\nu_{,i} \sim 2\nu x_i/(Ka_i^2) \quad \text{for } i = 1, 2 \text{ (no sum on } i)$$

while

$$\nu_{,3} \sim 2(\nu/K)^{1/2}$$

where

$$K = 1 - (x_1/a_1)^2 - (x_2/a_2)^2.$$

As $\nu \to 0$, we also have the result

$$\nu_{,i3} = \tilde{2}\delta_{i3}/K$$

where δ_{ij} is the kronecker delta.

We now return to the problem, and note that the global velocity distribution in terms of A_i and B_i is

$$u_i = U_i + \epsilon\{A_i a_i^2 I_i - A_j a_j^2 x_j \nu_{,i}/[(\alpha_j^2 + \nu)Q(\nu)] - B_i I_0 - x_k B_{k,i}/Q(\nu)\} \quad \text{(no sum on } i). \qquad [10]$$

In [10]

$$I_0 = \int_\nu^\infty Q^{-1}(\eta)\, d\eta \quad \text{and} \quad I_i = \int_\nu^\infty \{(a_i^2 + \eta)Q(\eta)\}^{-1}\, d\eta.$$

These integrals are available from tables of elliptic integrals (see for example Byrd & Friedman 1971). The no-slip condition is satisfied on the disc ($\nu = 0$) if A_j and B_j are chosen to be

$$B_j = -A_j = U_j/\{\epsilon(\bar{I}_0 + a_j^2\bar{I}_j)\}, \qquad j = 1, 2$$

$$A_3 = 0,\ B_3 = U_3/(\epsilon\bar{I}_0) \qquad [11]$$

where

$$\bar{I}_0 = \int_0^\infty Q(\eta)^{-1}\, d\eta \quad \text{and} \quad \bar{I}_j = \int_0^\infty \{(a_j^2 + \eta)Q(\eta)\}^{-1}\, d\eta.$$

It is observed here that the velocity field does not depend on any fluid property and more importantly that when $U_3 = 0$, $u_3 = 0$ on $x_3 = 0$, $\nu \neq 0$. The global pressure distribution P is obtained from [4] and [6] to be

$$P = 2\mu B_i I_{0,i}$$
$$= -4\epsilon\mu\, B_i \nu_{,i}/Q(\nu).$$

On the disc $\nu = 0$ and

$$P = -4\pi B_3 K^{-1/2}/(a_1 a_2) = -4U_3 K^{-1/2}/(a_1 a_2 \bar{I}_0)\,.$$

We note that when $U_3 = 0$ the pressure on the disc is zero. When $U_3 \neq 0$ the pressure distribution exhibits a square root singularity at the edge of the disc in confirmation of the finding of Olunloyo (1978).

The stress distribution is given by

$$\sigma_{ij} = -P\delta_{ij} + \mu[-\epsilon A_k a_k^2\{\delta_{ki}\nu_{,j}(a_k^2+\nu)^{-1}/Q(\nu) + 2\delta_{kj}\nu_{,i}(a_k^2+\nu)^{-1}/Q(\nu) + x_k\{\nu_{,i}(a_k^2+\nu)^{-1}/Q(\nu)\}_{,j} + x_k\{\nu_{,j}(a_k^2+\nu)^{-1}/Q(\nu)\}_{,i} - \epsilon B_k x_k\{(\nu_{,j}/Q(\nu))_{,i} - (\nu_{,i}/Q(\nu))_{,j}\}]\,. \qquad [12]$$

Noting that on $x_3 = 0$, $\nu \neq 0$

$$\nu_{,i3} = 2\delta_{i3}/[\nu\{x_1^2/(a_1^2+\nu)^2 + x_2^2/(a_2^2+\nu)^2 + (x_3/\nu)^2\}]\,,$$

we establish from [12] that

$$\sigma_{3i} = 0 \qquad i = 1, 2, 3$$

for points on $x_3 = 0$ which are not on the disc. This suggests that the plane $x_3 = 0$ acts as a stationary free surface bordered by a vacuum. If the fluid on one side of this plane ($x_3 < 0$ say) is replaced by another fluid which moves co-currently with the original fluid all the conditions of the problem will remain satisfied in the original fluid and velocity and stresses will be continuous across this plane. Moreover, $u_3 = 0$ on this plane when $U_3 = 0$.

Lastly, we note that on the disc ($\nu = 0$)

$$\sigma_{33} = -P \quad \text{and} \quad \sigma_{3i} = -4\mu A_i \epsilon K^{-1/2}/(a_1 a_2) \quad i = 1, 2.$$

3.1 *Lift and drag forces on disc in two-phase flow*

If the half space $x_3 > 0$ is occupied by a fluid with coefficient of viscosity μ_1 and $x_3 < 0$ is occupied by another fluid whose coefficient of viscosity is μ_2 and if both fluids move with uniform free stream velocity U_i about an elliptic disc situated on the plane $x_3 = 0$, the lift force on the disc is

$$F_3 = -\int\int_{s_1+s_2} P\,\mathrm{d}s$$

where s_1 and s_2 are the areas of the two faces of the disc.

$$F_3 = -4U_3(\mu_1+\mu_2)/a_1 a_2 \bar{I}_0 \int\int_0 K^{-1/2}\,\mathrm{d}s = -8\pi(\mu_1+\mu_2)U_3/\bar{I}_0\,. \qquad [13]$$

Similarly the drag force on the disc is

$$F_{12} = \int\int_{s_1+s_2} (-\sigma_{31}^2 + \sigma_{32}^2)^{1/2}\,\mathrm{d}s = 8\pi(\mu_1+\mu_2)\{U_1^2/(\bar{I}_0 + a_1^2\bar{I}_1)^2 + U_2^2/(I_0 + a_2^2 I_2)^2\}^{1/2}. \qquad [14]$$

The results of Olunloyo for a circular disc and that deduced from Oberbeck's result (see Moore 1964) can easily be recovered by setting $a_1 = a_2 = a$, the radius of the disc, in [13] and [14].

4. DISC IN LINEAR SHEAR FLOW

If in the absence of the disc the fluid velocity in the eventual position of the disc is

$$U_i^\infty = U_{ik}x_k \quad \text{where } U_{ii} = 0,$$

the potential functions which have the desired structure necessary for the satisfiation of all the conditions of the problem are

$$\psi_j = B_{ij}\bar{\psi}_i, \qquad i = 1, 2, \qquad j = 1, 2, 3, \tag{15}$$

$$\psi_0 = A_{ij}(\chi_{i,j} - x_j\bar{\psi}_i) \tag{16}$$

where

$$\chi_i = \int\int_s \{\xi_i J(\xi_1, \xi_2)R\}\, d\xi_1\, d\xi_2,$$

$$\bar{\psi}_i = \int\int_s \{\xi_i J(\xi_1, \xi_2)/R\}\, d\xi_1\, d\xi_2,$$

$$J(\xi_1, \xi_2) = \{1 - (\xi_1/a_1)^2 - (\xi_2/a_2)^2\}^{-1/2}$$

and A_{ij} and B_{ij} are constant tensors.

From [3] we obtain the expression for velocity to be

$$\bar{u}_n = U_{jn}x_j + A_{ji}\chi_{i,jn} - B_{jn}\bar{\psi}_j + B_{ji}x_i\bar{\psi}_{j,n} - A_{nj}\bar{\psi}_j - A_{ji}x_j\bar{\psi}_{i,n}. \tag{17}$$

The terms appearing in [17] can be deduced from the results given by Dyson and we have

$$\begin{aligned}\chi_{i,jn} = \epsilon a_i^2[&(x_j\delta_{in} + x_i\delta_{jn})\{2I_i - a_m^2\bar{I}_{jim} - 2a_i^2\bar{I}_{iij} + 2\bar{I}_{ij} \\ &+ 2a_j^4 I_{ijj} - 4a_j^2 I_{iji}\} + x_n\delta_{ij}\{2I_i - a_m^2\bar{I}_{inm} - 2a_i^2\bar{I}_{iin} \\ &+ 2a_i^2\bar{I}_{in} + 2a_n^4 I_{inn} - 4a_n^2 I_{in}\} + \delta_{ij}\{x_m x_m(I_{i,n} \\ &+ a_m^4 I_{imm,n} - 2a_m^2 I_{im,n}) + x_3^2 I_{i,n}\} + 2x_i x_j \\ &\times \{a_j^4 I_{ijj,n} - a_i^2\bar{I}_{iij,n} + I_{i,n} + \bar{I}_{ji,n} - 2a_j^2 I_{ij,n}\} \\ &+ \{x_3^2 x_i I_{i,j} - \nu^2 x_i x_m x_m I_{imm,j}\}_{,n}].\end{aligned} \tag{18}$$

Note that there is no sum on i, j, and n and that subscript m ranges from 1 to 2. Moreover, $a_3 = 0$.

$$\left.\begin{aligned}\bar{\psi}_i &= \epsilon a_i^2 x_i I_i \\ \bar{\psi}_{i,n} &= \epsilon(a_n^2 I_n + a_i^2 x_i I_{i,n})\end{aligned}\right] \text{(no sum)} \tag{19}$$

where

$$\left.\begin{aligned}I_{ijm} &= \int_\nu^\infty \{(a_i^2 + \eta)(a_j^2 + \eta)(a_m^2 + \eta)Q(\eta)\}^{-1}\, d\eta \\ \bar{I}_{ijm} &= \int_\nu^\infty \eta\{(a_i^2 + \eta)(a_j^2 + \eta)(a_m^2 + \eta)Q(\eta)\}^{-1}\, d\nu\end{aligned}\right]. \tag{20}$$

On invoking the no slip boundary condition[5] on the disc, the following equations result for A_{jn} and B_{jn}.

For n = 1, 2

$$B_{nj} + B_{jn} - A_{nj} - A_{jn} = 0, \quad n \neq j,$$
$$B_{11} = B_{11}; \quad B_{22} = A_{22}, \qquad [21]$$

and

$$\epsilon a_n^2[\{2\bar{I}_n - a_m^2\bar{\bar{I}}_{jnm} - 2a_n^2\bar{\bar{I}}_{nnj} + 2\bar{\bar{I}}_{nj} + 2a_j^4\bar{I}_{njj}\}(A_{jn} + A_{nn})$$
$$- 4\bar{I}_{nj}(a_j^2A_{jn} + a_n^2A_{nn})] + \epsilon a_j^2A_{jn}\{\bar{I}_j - a_m^2\bar{\bar{I}}_{njm} - 2a_j^2\bar{\bar{I}}_{ljn}$$
$$+ 2\bar{\bar{I}}_{nj} + 2a_n^2\bar{I}_{jnn} - 4a_n^2\bar{I}_{jn}\} - \epsilon B_{nj}a_j^2\bar{I}_j + U_{nj} = 0 .$$
$$\text{(no sum on } n \text{ and } j) \qquad [22]$$

$\bar{I}_{jnm}$ and $\bar{\bar{I}}_{jnm}$ etc. are evaluated by replacing the lower limits of the integrals in [20] with zero.

For n = 3

Equations [21] still hold while in place of [22] we have

$$\epsilon a_j^2[A_{j3}\{\bar{I}_j - a_m^2\bar{\bar{I}}_{3jm} - 2a_j^2\bar{\bar{I}}_{jj3} + 2\bar{\bar{I}}_{3j}\} - B_{j3}] + U_{j3} = 0 \quad \text{(no sum on } j).$$

Again it is easy to verify that the normal velocity is zero in all of the plane $x_3 = 0$.

The pressure distribution is given by

$$P = 2\mu B_{ij}a_j^2\epsilon(\delta_{ij}I_j + x_jI_{j,i}). \qquad [23]$$

The second term in [23] exhibits a square root singularity at the edge of disc. The stress distribtuion can be obtained from [3], [15] and [16]. The resulting expression is rather long and will not be reproduced here. We however give a few salient features of the stress distribution on the plane $x_3 = 0$. On the disc ($x_3 = 0$, $\nu = 0$)

$$\sigma_{33} = -P.$$

Also $\sigma_{3i} = 0$ on $x_3 = 0$, $\nu \neq 0$ for $i = 1, 2, 3$.

Moreover on the disc ($\nu = 0$) the tangential stress distribution is antisymmetric with respect to the axes of the disc. It therefore follows that the drag on the disc in shear flow is zero. This result holds not only for the elliptic disc but for any disc which is symmetrical about axes in its plane. Finally the lift force on a disc at the planar interface of a two-phase fluid in shear flow is

$$F_3 = -\int\int_{s_1+s_2} P\,\mathrm{d}s$$
$$= 2\pi a_1a_2(\mu_1 + \mu_2)\epsilon_1\{B_{11}I_1a_1^2 + B_{22}I_2a_2^2\}.$$

5. GENERALISATION TO GENERAL POLYNOMIAL FLOW

The procedure adopted in the preceeding section can actually be generalised to the case where the free stream velocity is a general polynomial function of coordinates x_1 and x_2.

Let the velocity of the fluid in the absence of the disc be

$$u_i^\infty = U_{ijk\ldots r}\xi_j\xi_j \ldots \xi_r \qquad [24]$$

where $j, k, \ldots, r = 1, 2$; $i = 1, 2, 3$.

Let the velocity field U_i^{∞} and the associated pressure field satisfy [1] and [2]. Then the potential functions which possess the requisite structure for the flow when the disc is now placed in the plane $x_3 = 0$ are

$$\psi_i = A_{ijk\ldots r}\int\int_s \{\xi_j\xi_k\ldots\xi_r J(\xi_1, \xi_2)/R\}\,d\xi_1\,d\xi_2\,.$$

$$\psi_0 = B_{ijk\ldots r}(\chi_{jk\ldots r,i} - x_i\bar{\psi}_{jk\ldots r})$$

where

$$\chi_{jk\ldots r} = \int\int_s \{\xi_j\xi_k\ldots\xi_r J(\xi_1, \xi_2)R\}\,d\xi_1\,d\xi_2$$

and

$$\bar{\psi}_{jk\ldots r} = \psi_i A^{-1}_{ijk\ldots r}.$$

The velocity components are

$$u_i = U_{ijk\ldots r}x_j x_k\ldots x_r + \{\psi_0 + x_m\psi_m\}_{,i} - 2\psi_i\,.$$

Satisfaction of the no slip conditon gives rise to two matrix equations of the form

$$C_{ijk\ldots r}x_i x_j x_k\ldots x_r = 0\,,$$

$$D_{ijk\ldots r}x_i x_j x_k\ldots x_r = 0 \qquad [25]$$

where C and D are matrices whose elements depend on those of A, B and U. Equation [25] are solved to obtain expressions for the elements of A and B in terms of those of U. The pressure and stress distributions are obtained by using these potentials in [4] and [3] respectively. The two-phase solution is obtained by combining the single-phase solution for each of the phases without any further modification.

6. CONCLUDING REMARKS

A method has been presented for solving the problem of an elliptic disc moving at the interface (assumed planar) of two immiscible fluids. It is demonstrated, by working out the solution for uniform and shear flows, that there are solutions which satisfy the conditions of continuity of normal and tangential components of stress and velocity as well as that of zero normal interfacial velocity. It is also shown that the flow fields described by these solutions are such that there is no interaction between the two immiscible fluids. The result reported by Olunloyo, that the pressure distribution exhibits a singularity of the square root type is here confirmed even though their method of solving the problem differs somewhat from ours. The values of the drag and lift forces which we obtain for the elliptic disc in uniform two-phase flow reduces to those of Olunloyo for a circular disc on taking the appropriate limit. A procedure is outlined for extending the method to the case where the free stream two-phase fluid velocity is a general polynomial function of the coordinates x_1 and x_2. Finally we remark that the method can easily be applied to disc geometries for which the resulting surface integrals can be evaluated with the aid of potential theory. The problem of a disc moving inside one phase of a two-phase fluid medium will be addressed in a future paper.

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